

# THERE EXIST NO MINIMALLY KNOTTED PLANAR SPATIAL GRAPHS ON THE TORUS

SENJA BARTHEL

**ABSTRACT.** We show that all nontrivial embeddings of planar graphs on the torus contain a nontrivial knot or a nonsplit link. This is equivalent to showing that no minimally knotted planar spatial graphs on the torus exist that contain neither a nontrivial knot nor a nonsplit link all of whose components are unknots.

## 1. INTRODUCTION

All considered graphs are undirected finite graphs and we will work in the piecewise linear category. A **graph embedding** is an embedding  $f : G \rightarrow S^3$  of a graph  $G$  in  $S^3$  up to ambient isotopy and the corresponding **spatial graph**  $\mathcal{G}$  is the image of this embedding. A graph  $G$  is **planar** if there exists an embedding  $f : G \rightarrow S^2$ . An embedding  $f : G \rightarrow S^3$  is **trivial** if  $\mathcal{G}$  is contained in a 2-sphere embedded in  $S^3$ . Its image  $\mathcal{G}$  is a **trivial spatial graph**. A spatial graph  $\mathcal{G}$  is **minimally knotted** if  $\mathcal{G}$  is nontrivial but  $\mathcal{G} - e$  is trivial for every edge  $e$ . Some authors call minimally knotted spatial graphs **almost trivial**, **almost unknotted** or **Brunnian**. In this paper, a nontrivial **link** is a nonsplit link with at least two components.

Previous research on minimally knotted spatial graphs has been undertaken: The first example of a minimally knotted spatial graph was an embedding of a handcuff graph given by Suzuki [1]. Kawauchi [2], Wu [3] and Inaba and Soma [4] showed that every planar graph has a minimally knotted embedding. Ozawa and Tsutsumi [5] proved that minimally knotted embeddings of planar graphs are totally knotted. Especially minimally knotted  $\theta_n$ -graphs have generated some interest. Kinoshita [6] gave the first example of a minimally knotted  $\theta_3$ -graph (see Fig. 1) which Suzuki [7] generalised to give examples of minimally knotted  $\theta_n$ -graphs for all  $n \geq 3$ . Closely related are **ravels** which are nontrivial embeddings of  $\theta_n$ -graphs that contain no nontrivially knotted subgraph; this definition is equivalent to the one given by Farkas, Flapan and Sullivan [8]. The concept of ravels has been introduced by Castle, Evans and Hyde [9] as local entanglements that are not caused by knots or links and may lead to new topological structures in coordination polymers. A ravel in a molecule has been synthesized by Lindoy *et al* [10]. Castle, Evans and Hyde [11] conjectured the following:

**Conjecture** (Castle, Evans, Hyde [11]). All nontrivial embeddings of planar graphs on the torus include a nontrivial knot or a nonsplit link.

With Theorem 1 we prove that their conjecture is true. With **torus** we refer to an embedded torus in the 3-sphere  $S^3$  which may be nonstandardly embedded. A **standardly embedded torus** is a torus that bounds two solid tori in  $S^3$ . A nonstandardly embedded torus still bounds a solid torus in  $S^3$  by the Solid Torus Theorem [12].

**Theorem 1** (Knots and links existence). *Let  $G$  be a planar graph and  $f : G \rightarrow S^3$  be an embedding of  $G$  with image  $\mathcal{G}$ . If  $\mathcal{G}$  is contained in the torus  $T^2$  and contains no nontrivial knot nor a nonsplit link, then  $f$  is trivial.*

Since  $\theta_n$ -graphs are planar, it follows from Theorem 1 that on the torus there exist no minimally knotted embeddings of  $\theta_n$ -graphs with  $n > 2$ . This gives us the following:

**Corollary 1** (Ravels do not embed on the torus). Every nontrivial embedding of  $\theta_n$ -graphs on the torus contains a knot.

We conclude by showing that all assumptions of Theorem 1 are necessary. Explicit ambient isotopies that transform spatial graphs which fulfil the assumptions of Theorem 1 into the plane  $\mathbb{R}^2$ , are given in [13]. Another consequence of Theorem 1 that is stated in Remark 1 has been shown in [11] together with [14]: Nontrivial 3-connected and simple planar spatial graphs that are embedded on a torus are chiral. A graph is **simple** if it contains no loops and no multi-edges. It is **3-connected** if at least three vertices and their incident edges have to be deleted to decompose the graph or to reduce it to a single vertex. A spatial graph is **chiral** if it is not ambient isotopic to its mirror image.

**2.1. Outline of the proof.** The proof uses two theorems of Scharlemann, Thompson [15] and Ozawa, Tsutsumi [5]. We assume that the spatial graph  $\mathcal{G}$  we consider is given by an embedding  $f : G \rightarrow T^2$  of a planar graph  $G$  and furthermore that  $\mathcal{G}$  contains no nontrivially knotted or linked subgraph. We conclude that  $\mathcal{G}$  must be trivial. During the proof, we need the following two definitions:

**Definition 1.** An embedding  $f : G \rightarrow S^3$  of a graph  $G$  is **primitive**, if for each component  $G_i$  of  $G$  and any spanning tree  $T_i$  of  $G_i$ , the bouquet graph  $f(G_i)/f(T_i)$  obtained from  $f(G_i)$  by contracting all edges of  $f(T_i)$  in  $S^3$  is trivial.

**Definition 2.** An embedding  $f : G \rightarrow S^3$  of a graph  $G$  is **free**, if the fundamental group of  $S^3 - f(G)$  is free.

The argument of the proof is as follows: We start showing that the statement is true for nonstandardly embedded tori in Lemma 1. With Lemma 2 we argue that it is sufficient to consider connected graphs. Then we show in Lemma 3 that a bouquet graph on  $T^2$  either contains a nontrivial knot or is trivial. Since any connected spatial graph  $\mathcal{G}$  on  $T^2$  contracts to a bouquet graph on  $T^2$ , it follows that  $\mathcal{G}$  is primitive if it contains no nontrivial knot. By Theorem 2 we know that the restriction  $f|_{G'}$  is free for all connected subgraphs  $G'$  of  $G$ . By Lemma 2 together with Theorem 3 we conclude that  $\mathcal{G}$  is trivial.

## 2.2. Preparations for the proof.

**Lemma 1** (Nonstandardly embedded torus). Let  $\mathfrak{T}^2$  be a torus that is not standardly embedded. Any spatial graph  $\mathcal{G}$  that is embedded on  $\mathfrak{T}^2$  and that contains no nontrivial knot is trivial.

*Proof.* If the spatial graph  $\mathcal{G}$  contains a cycle that follows a longitude of the torus  $\mathfrak{T}^2$ , this cycle is knotted since  $\mathfrak{T}^2$  itself is knotted. Therefore, no such subgraph of  $\mathcal{G}$  can exist and we find a meridian  $m$  of  $\mathfrak{T}^2$  that has no intersection with  $\mathcal{G}$ . This shows that  $\mathcal{G}$  is embedded in the twice punctured sphere  $\mathfrak{T}^2 - m \simeq S^2 - \{p_1, p_2\}$ . Therefore,  $\mathcal{G}$  is trivial.  $\square$

It follows from Lemma 1 that the statement of Theorem 1 is true for nonstandardly embedded tori. Therefore, we only consider the standardly embedded torus  $T^2$  from now on which saves us from considering different cases.

**Lemma 2** (Connectivity Lemma). The image  $\mathcal{G}$  of an embedding  $f : G \rightarrow T^2 \subset S^3$  of a graph  $G$  with  $n > 1$  connected components on the standard torus  $T^2$  contains either a nonsplit link, or contains no nonsplit link and decomposes into  $n$  disjoint components of which at least  $n - 1$  components are trivial.

*Proof.* Take any connected component  $f(G_i)$  of the embedding  $f(G)$  on the torus  $T^2$ . The complement of  $f(G_i)$  in the torus (without considering the rest of the spatial graph  $f(G - G_i)$ ) is a collection of pieces that can be the punctured torus, discs and essential annuli without boundaries. (An essential annulus contains a simple closed curve that does not bound a disc in the torus.)

In the case that the complement of  $f(G_i)$  in  $T^2$  includes the punctured torus,  $f(G_i)$  is trivial and splits from the other components.

If the complement of  $f(G_i)$  in  $T^2$  is only a collection of discs, then all other components of  $f(G)$  lie in one of those discs and therefore are trivial and the graph is split. ( $f(G_i)$  might or might not contain a nonsplit link.)

In the case that the complement of  $f(G_i)$  in  $T^2$  includes an essential annulus  $A$ , it is possible that other components of  $G$  are embedded in this annulus. A component  $G_j$  might be embedded in the annulus in two ways: Either the complement of  $f(G_j)$  in  $A$  includes a punctured annulus and therefore  $f(G_j)$  is trivial and splits from the rest of the spatial graph  $f(G - G_j)$ . Or  $A - f(G_j)$  contains two annuli. The annulus  $A$  has one type of an essential curve  $c$  running inside it;  $c$  is parallel to the boundary curves of  $A$ . In the case that  $A - f(G_j)$  contains two annuli, a subgraph of  $f(G_j)$  must be deformable to be parallel to  $c$ . If  $c$  is a meridian or a preferred longitude of  $T^2$ , both components  $f(G_i)$  and  $f(G_j)$  are split and trivial since the torus is a standard torus. If  $c$  is neither a meridian nor a longitude of  $T^2$ ,  $f(G_i)$  and  $f(G_j)$  are nonsplittably linked.  $\square$

**Lemma 3** (Bouquet Lemma). The image  $\mathcal{B}$  of an embedding  $f : B \rightarrow T^2 \subset S^3$  of a connected bouquet graph  $B$  on the torus  $T^2$  either contains a nontrivial knot or is trivial.

*Proof.* All cycles of a spatial bouquet graph  $\mathcal{B}$  on  $T^2$  that contains no nontrivial knot are the unknot by assumption. The unknot on the torus can take the following forms:

- (1)  $T(0, 0)$  loops that bound a disc in  $T^2$  (trivial elements in  $\pi_1(T^2)$ ),

- (2)  $T(0, 1)$  meridional loops,
- (3)  $T(1, 0)$  longitudinal loops,
- (4)  $T(1, n)$  loops or alternatively  $T(n, 1)$  loops,  $n \geq 1$

Loops of type (1) do not contribute to nontriviality of  $\mathcal{B}$ .

If  $\mathcal{B}$  has loops of the types (1), (2) and (3) only, it is trivial.

If  $\mathcal{B}$  has loops of type (4), there are – beside the loops  $T(0, 0)$  – only three types of loops simultaneously embeddable on the torus without self-intersections:  $T(0, 1)$ ,  $T(1, n)$  and  $T(1, n + 1)$  (respectively  $T(1, 0)$ ,  $T(n, 1)$  and  $T(n + 1, 1)$ ). This can easily be confirmed by applying the formula of Rolfsen’s exercise 2.7 [16]: If two torus knots  $T(p, q)$  and  $T(p', q')$  intersect in one point transversally, then  $pq' - qp' = \pm 1$ . Such a bouquet is trivial.  $\square$

**Theorem 2** (Ozawa and Tsutsumi’s freeness criterion [5]). *An embedding  $f$  of a graph  $G$  in  $S^3$  is primitive if and only if the restriction  $f|_{G'}$  is free for all connected subgraphs  $G'$  of  $G$ .*

**Theorem 3** (Scharlemann and Thompson’s planarity criterion [15]). *An embedding  $f(G) \rightarrow S^3$  of a graph  $G$  is trivial if and only if*

- (a)  $G$  is planar and
- (b) for every subgraph  $G' \subset G$ , the restriction  $f|_{G'}$  is free.

**2.3. The proof.** We are now ready to prove Theorem 1 and Corollary 1:

*Proof.* (of Theorem 1). It follows from Lemma 1 that the statement of Theorem 1 is true for nonstandardly embedded tori. Therefore, we assume that  $\mathcal{G}$  is embedded in the standard torus  $T^2$ . By Lemma 2 (Connectivity Lemma) we can assume that  $G$  is connected. Any connected spatial graph contracts to a spatial bouquet graph  $\mathcal{B}$  if a spanning tree  $T$  is contracted in  $S^3$ . If the spatial graph is embedded in a surface, edge contractions can be realised in the surface. It follows that a connected spatial graph  $\mathcal{G}$  which is embedded in the torus  $T^2$ , contracts to a bouquet graph  $B$  which also is embedded in  $T^2$  if a spanning tree is contracted. Since  $\mathcal{G}$  contains no nontrivial knot by assumption,  $B$  also contains no nontrivial knot. It follows from Lemma 3 that a bouquet graph  $B$  that contains no nontrivial knot on the torus  $T^2$  is trivial. Therefore, any bouquet graph  $B = f(G)/f(T)$  which is obtained from  $f(G)$  by contracting all edges of  $f(T)$  in  $S^3$  is trivial and  $f$  is primitive by definition. By Theorem 2, the restriction  $f|_{G'}$  is free for all connected subgraphs  $G'$  of  $G$ . Then Lemma 2 ensures that the restriction  $f|_{G'}$  is free for all subgraphs  $G'$  of  $G$  since  $\mathcal{G}$  contains no nonsplit link by assumption. As  $G$  is planar by assumption, it follows from Theorem 3 that  $f(G)$  is trivial.  $\square$

*Proof.* (of Corollary 1). As there exist no pair of disjoint cycles in a  $\theta_n$ -graph, such a graph does not contain a nonsplit link. Since  $\theta_n$ -graphs are planar, the statement of the corollary follows directly from Theorem 1.  $\square$

It has been shown in [11] together with [14] that every nontrivial embedding of a simple 3-connected spatial graph on the torus that contains a nontrivial knot or a nonsplit link is chiral. The following Remark 1 is therefore a consequence of Theorem 1.

**Remark 1** (Chirality). Nontrivial embeddings of simple 3-connected planar graphs in the torus are chiral.

**2.4. All assumptions that have been made are necessary.** This can be seen by considering the following examples:

- There exist nontrivial embeddings on  $T^2$  that contain neither a nontrivial knot nor a nonsplit link. Those are embeddings of graphs which are not planar.  
*Examples:*  $K_{3,3}$  and  $K_5$  embedded as shown left in Fig. 1.
- There exist nontrivial embeddings of planar graphs that contain neither a nontrivial knot nor a nonsplit link. Those are not embedded in the torus.  
*Examples:* Kinoshita-theta curve (middle in Fig. 1) and every ravel.
- There exist nontrivial embeddings of planar graphs on  $T^2$ .  
*Examples:* Spatial graphs that are subdivisions of nontrivial torus knots with  $n > 0$  vertices and  $n$  edges (right in Fig. 1).

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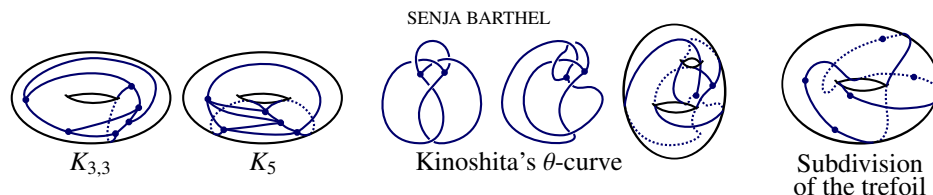


FIGURE 1. All assumptions are necessary.

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DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON, SW7 2AZ, UNITED KINGDOM  
 E-mail address: s.barthel11@imperial.ac.uk